Kuwait University	**Math. 101**	**July 24, 2010**
Dept. of Math.	**Second Midterm Exam**	**Duration 90 Minutes**

Calculators and mobile phones are not allowed

Answer the following questions.

1. **(3 Points)** Let $y = \frac{1}{2}\sqrt{4v^2 - 4v + 1}$ and $v = \cos(x^2) + \dfrac{1}{2 - \tan x}$. Find $\dfrac{dy}{dx}$ at $x = 0$.

2. **(3 Points)** Find an equation of the normal line to the graph of the equation
$$(x^2 + y^2)^2 = 5xy + 15$$
at the point $P(1,2)$.

3. **(3 Points)** Find the value of dy if
$$y = x^2 + \frac{1}{\pi}\sec^2\left(\frac{\pi}{12}x\right)$$
and x changes from 3 to 3.03.

4. **(3 Points)** A point $P(x,y)$ moves on the curve $y = \sqrt{x}$. If S is the distance between the point P and the origin, then find $\dfrac{dS}{dt}$ when $x = 1$ and $\dfrac{dx}{dt} = 2$.

5. **(3 Points)** Use Rolle's Theorem to show that the equation
$$4x^3 + 4x + 3 = 0$$
cannot have more than one real root.

6. **(10 Points)** Let $f(x) = \dfrac{6x - 6}{x^2}$.

   a. Show that $f'(x) = \dfrac{6(2 - x)}{x^3}$ and $f''(x) = \dfrac{12(x - 3)}{x^4}$.

   b. Find the intervals on which f is increasing or decreasing, and find the local extrema of f, if any.

   c. Find the intervals on which the graph of f is concave up or concave down, and find the points of inflection, if any.

   d. Find the vertical and horizontal asymptotes of the graph of f, if any.

   e. Sketch the graph of f.

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1. At $x = 0$, $v = \frac{3}{2}$: $\frac{dy}{dv} = \frac{2v-1}{\sqrt{4v^2-4v+1}}$, $\frac{dv}{dx} = -2x\sin x^2 + \frac{\sec^2 x}{(2-\tan x)^2}$

$$\left[\frac{dy}{dx}\right]_{x=0} = \left[\frac{dy}{dv}\right]_{v=\frac{3}{2}} \left[\frac{dv}{dx}\right]_{x=0} = (1)(\tfrac{1}{4}) = \boxed{\tfrac{1}{4}}$$

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2. $2(x^2+y^2)(2x+2yy') = 5(y+xy') \Rightarrow m = [y']_{(1,2)} = -\frac{2}{7} \Rightarrow m_\perp = \frac{7}{2}$

Equation of normal line : $\boxed{(y-2) = \frac{7}{2}(x-1)}$

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3. $y = f(x) = x^2 + \frac{1}{\pi}\sec^2\left(\frac{\pi}{12}x\right)$

$$dy = f'(x)dx = \left[2x + \frac{1}{6}\sec^2\left(\frac{\pi}{12}x\right)\tan\left(\frac{\pi}{12}x\right)\right]dx$$

when $x = 3$ and $dx = \Delta x = 0.03$, this becomes

$$dy = \left[6 + \tfrac{1}{6}(2)(1)\right](0.03) = \boxed{0.19}$$

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4. $S = \sqrt{x^2+y^2} = \sqrt{x^2+x}$, $\frac{dS}{dt} = \left(\frac{dS}{dx}\right)\left(\frac{dx}{dt}\right) = \left(\frac{2x+1}{2\sqrt{x^2+x}}\right)\left(\frac{dx}{dt}\right)$,

when $x = 1$ and $\frac{dx}{dt} = 2$: $\frac{dS}{dt} = \left(\frac{3}{2\sqrt{2}}\right)(2) = \boxed{\frac{3}{\sqrt{2}}}$

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5. Let $f(x) = 4x^3+4x+3 \Rightarrow f'(x) = 12x^2+4$

Suppose that $f(x) = 0$ has 2 different real roots c_1, c_2 with $c_1 < c_2$.

So $f(c_1) = f(c_2) = 0$.

Since f is a polynomial, it is continuous on $[c_1, c_2]$ and differentiable on (c_1, c_2).

By Rolle's Theorem, there is a number k in (c_1, c_2) such that $f'(k) = 0$.

But $f'(x) = 12x^2+4 \geq 4$ for all x, so $f'(x)$ can never be zero. This contradiction shows that the equation cannot have more than one real root.

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6. $f(x) = \dfrac{6x-6}{x^2}$: Domain of $f = \mathbb{R} - \{0\}$. The point $(1,0)$ lies on the graph of f.

(a) $f(x) = \dfrac{6x-6}{x^2} \Rightarrow f'(x) = \dfrac{6x^2 - 2x(6x-6)}{x^4} = \boxed{\dfrac{6(2-x)}{x^3}}$

$f'(x) = \dfrac{6(2-x)}{x^3} \Rightarrow f''(x) = \dfrac{-6x^3 - 3x^2(12-6x)}{x^4} = \boxed{\dfrac{12(x-3)}{x^4}}$

(b)

Interval	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
sign of $f'(x)$	$-$	$+$	$-$
conclusion / f	↘	↗	↘

$\boxed{f(2) = \tfrac{3}{2} \text{ is a local maximum}}$

(c)

Interval	$(-\infty, 0)$	$(0, 3)$	$(3, \infty)$
sign of $f''(x)$	$-$	$-$	$+$
conclusion / graph of f	CD	CD	CU

$\boxed{\left(3, \tfrac{4}{3}\right) \text{ is a point of inflection}}$

(d) $\lim_{x \to 0^{\pm}} f(x) = -\infty \Rightarrow \boxed{x = 0 \text{ is V.A}}$ $\lim_{x \to \pm\infty} f(x) = 0 \Rightarrow \boxed{y = 0 \text{ is H.A}}$

(e)